ASSIGNMENT 2

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Part (a): Group 1 Functions

1. y = e^x : This shows exponential growth. As x gets larger, y increases very quickly. For negative x , y gets close to zero.

2. y = e^{-x} : This shows exponential decay. As x gets larger, y decreases towards zero, but for negative x , it grows quickly.

3. y = 8^x : This grows even faster than e^x , since the base (8) is bigger.

4. y = 8^{-x} : This decays faster than e^{-x} , because the base is larger.

Part (b): Group 2 Functions

1. y = 0.9^x : This shows slow decay. As x increases, y gradually decreases.

2. y = 0.6^x : Faster decay than 0.9, meaning it decreases more quickly as x increases.

3. y = 0.3^x : Decays even faster than the previous two.

4. y = 0.1^x : This decays the quickest, approaching zero very fast as x increases.

Key Points:

• Exponential growth: Functions like e^x and 8^x grow very fast as x gets larger.

• Exponential decay: Functions like e^{-x} , 0.9^x , and others get smaller as x increases, with some decaying faster than others based on their base.

**Question 2**

For a function f(x), the derivative can be approximated by the difference quotient:  
(f(x+h) - f(x)) / h  
This expression gives an approximation of the rate of change of f(x) as x changes by a small amount h. As h approaches 0, this difference quotient becomes the exact derivative.

**Given Function**

We are given:  
f(x) = 10^x.  
The goal is to prove that:  
(f(x+h) - f(x)) / h = 10^x \* (10^h - 1) / h.  
This expression represents the difference quotient for f(x) = 10^x, and we aim to verify it through calculations and plotting in Excel.

**Step-by-Step Derivation**

**1. Calculate f(x+h)**

For f(x) = 10^x, we can calculate f(x+h):  
f(x+h) = 10^(x+h) = 10^x \* 10^h.

**2. Apply the Difference Quotient Formula**

The difference quotient is given by:  
(f(x+h) - f(x)) / h.  
Substitute f(x+h) and f(x) into this formula:  
(10^x \* 10^h - 10^x) / h.  
Factor out 10^x:  
10^x \* (10^h - 1) / h.  
Thus, we have:  
(f(x+h) - f(x)) / h = 10^x \* (10^h - 1) / h.  
This proves the required expression.

**In Excel :**

**Question 3**

Comparing the Functions f(x) = x^5 and g(x) = 5^x

We are tasked with comparing two functions:

• f(x) = x^5 (a polynomial function)

• g(x) = 5^x (an exponential function)

We need to determine which function grows more rapidly as x becomes large. We will both plot the functions and analyze their growth rates mathematically.

Step 1: Understanding the Growth of Polynomial and Exponential Functions

1. Polynomial Growth:

f(x) = x^5 is a polynomial function. As x increases, f(x) grows at a rate proportional to x^5. For small values of x, this function grows steadily, but eventually, its growth is limited compared to exponential functions as x becomes large.

2. Exponential Growth:

g(x) = 5^x is an exponential function. Exponential functions grow much more quickly than polynomial functions for large values of x. This is because, for every additional increase in x, the function is multiplied by 5, causing rapid increases.

Step 2: Mathematical Proof of Growth Rate Comparison

To compare their growth, we analyze their behavior as x approaches infinity.

Growth of f(x) = x^5:

As x approaches infinity, f(x) grows indefinitely, but its rate of increase is constrained by the fact that it is only raised to the power of 5. It can be written as:

f(x) = x^5

Growth of g(x) = 5^x:

Exponential functions grow significantly faster. As x approaches infinity, g(x) increases exponentially. It can be written as:

g(x) = 5^x

Step 3: Proving Which Grows Faster Using Limits

To prove which function grows faster, let’s take the limit of the ratio of the two functions as x approaches infinity:

Limit as x approaches infinity of (x^5) / (5^x).

We apply L’Hopital’s Rule by repeatedly differentiating the numerator and denominator. After differentiating five times, the numerator becomes 0, while the denominator remains an exponential function, yielding:

Limit as x approaches infinity of 0 / 5^x = 0.

Thus, the exponential function g(x) = 5^x grows faster than the polynomial function f(x) = x^5 for large values of x.

Step 4: Graphical Verification in Excel

Conclusion:

While both functions grow as x increases, the exponential function g(x) = 5^x grows far more rapidly than the polynomial function f(x) = x^5 for large values of x. This can be proven mathematically and confirmed through graphical analysis in Excel.

**Question 4**

Proving f(x) is an Odd Function

Given Function:

The function is defined as:

f(x) = (1 - e^(1/x)) / (1 + e^(1/x))

Definition of an Odd Function:

A function f(x) is considered odd if it satisfies the condition:

f(-x) = -f(x)

for all x in the domain of the function.

Step 1: Find f(-x)

Substitute -x into the function:

f(-x) = (1 - e^(1/(-x))) / (1 + e^(1/(-x))) = (1 - e^(-1/x)) / (1 + e^(-1/x))

Step 2: Simplify f(-x)

Using the property of exponents e^(-1/x) = 1 / e^(1/x), rewrite the expression:

f(-x) = (e^(1/x) - 1) / (e^(1/x) + 1)

Step 3: Compare f(-x) to -f(x)

Now, compare this to -f(x). The original function is:

f(x) = (1 - e^(1/x)) / (1 + e^(1/x))

Multiply f(x) by -1:

-f(x) = -( (1 - e^(1/x)) / (1 + e^(1/x)) ) = (e^(1/x) - 1) / (e^(1/x) + 1)

Thus, we observe that:

f(-x) = -f(x)

In Excel :

Conclusion:

Since f(-x) = -f(x), we have proven that the function f(x) = (1 - e^(1/x)) / (1 + e^(1/x)) is an odd function.

**Question 5**

Understanding the Parametrized Function

We are given the function:

f(x) = 1 / (1 + a \* e^(bx))

where a > 0. The task is to examine how the graph of the function changes when the parameters b and a vary.

Part (a): Effect of Changing b

When b changes while a remains constant, the shape of the curve changes. b controls the rate of growth or decay in the function.

• For positive values of b: The function f(x) approaches zero faster as x increases. The higher the value of b, the steeper the decline in f(x) for positive values of x. The graph will show rapid decay for larger values of b.

• For negative values of b: The function increases more slowly as x becomes more negative. As b becomes more negative, the graph becomes flatter, and the rate of decay becomes less steep.

Part (b): Effect of Changing a

When a changes while b remains constant, a affects the height and position of the curve.

• For larger values of a: The curve shifts downward as the value of a increases, since the term a \* e^(bx) becomes larger, pulling the function values closer to zero. This causes the function to decay faster.

• For smaller values of a: The curve shifts upward as a decreases, meaning the function takes longer to decay. The graph will be flatter for smaller values of a, as the exponential term has less influence.

In excel :

Conclusion:

By adjusting b and a, the curve’s steepness and position change significantly. The parameter b controls the rate of decay, while a shifts the curve up or down.

**Question 6**

Problem: Find the Inverse Function and Plot

We are given the function:

g(x) = x^6 + x^4, where x ≥ 0.

Step 1: Find the Inverse Function g^(-1)(x)

To find the inverse function g^(-1)(x), we need to express x in terms of y, where y = g(x) = x^6 + x^4. Unfortunately, this equation cannot be solved algebraically using elementary methods to express x directly in terms of y. Therefore, finding a precise symbolic form for g^(-1)(x) is not feasible for this equation.

Instead, we can approach this numerically. You can use methods like Newton’s method or other numerical techniques to find approximate values for g^(-1)(x), but it does not have a straightforward algebraic expression.

Step 2: Plot y = g(x), y = x, and y = g^(-1)(x)

Since finding the exact inverse function algebraically is difficult, you can plot the graphs of these functions numerically:

1. y = g(x): This is the given function g(x) = x^6 + x^4.

2. y = x: This is the line where the output equals the input, and the graph will be a 45-degree diagonal line.

3. y = g^(-1)(x): This represents the inverse function. You can approximate the values for the inverse function by solving numerically or graphically by reflecting the graph of g(x) across the line y = x.

Numerical or Graphical Approach:

1. Use a spreadsheet or graphing tool to input a range of values for x (e.g., from 0 to 10).

2. Calculate the corresponding values of g(x) = x^6 + x^4.

3. Plot y = g(x), y = x, and reflect the graph across the line y = x to get an approximate visual for g^(-1)(x). This will give you an understanding of how the inverse function behaves.

In Excel :

Conclusion:

While finding an algebraic expression for g^(-1)(x) is difficult due to the nature of the function, it can be approximated numerically or graphically. By plotting the graphs of y = g(x), y = x, and y = g^(-1)(x), you can visualize the relationship between the function and its inverse.

**Question 7**

Problem 7: Camera Flash Capacitor Charging

We are given the function for the charge stored in the capacitor after a camera flash, which is:

Q(t) = Q0(1 - e^(-t/a))

Where:

• Q0 is the maximum charge capacity.

• t is time in seconds.

• a is a constant that affects the charging rate.

Part (a): Finding the Inverse Function

To find the inverse of the function Q(t) = Q0(1 - e^(-t/a)), we need to express t in terms of Q.

1. Start with the given equation:

Q = Q0(1 - e^(-t/a))

2. Divide both sides by Q0:

Q / Q0 = 1 - e^(-t/a)

3. Subtract 1 from both sides:

(Q / Q0) - 1 = -e^(-t/a)

4. Multiply both sides by -1 to eliminate the negative sign on the right:

1 - (Q / Q0) = e^(-t/a)

5. Take the natural logarithm (ln) of both sides to remove the exponential:

ln(1 - Q / Q0) = -t/a

6. Solve for t:

t = -a \* ln(1 - Q / Q0)

Thus, the inverse function is:

t = -a \* ln(1 - Q / Q0)

Meaning of the Inverse Function:

This inverse function represents the time it takes for the capacitor to store a charge Q, given that the maximum charge capacity is Q0 and the constant rate parameter is a.

Part (b): How Long to Reach 90% of the Charge Capacity

To find the time it takes to charge the capacitor to 90% of its maximum capacity, we set Q = 0.9 \* Q0 and solve for t.

1. Substitute Q = 0.9 \* Q0 into the inverse function:

t = -a \* ln(1 - (0.9 \* Q0 / Q0))

2. Simplify the equation:

t = -a \* ln(1 - 0.9)

t = -a \* ln(0.1)

3. Using the value a = 2:

t = -2 \* ln(0.1)

4. Calculate the result:

t ≈ -2 \* (-2.3026)

t ≈ 4.6052 seconds

Therefore, it takes approximately 4.6 seconds to charge the capacitor to 90% of its maximum capacity when a = 2.

In excel